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Mathematical model of track cycling: the individual pursuit

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Abstract

Mathematical modelling has been used to predict the performance of cyclists for a number of years. The aim of this study was to develop a mathematical model for Individual Pursuit (IP) cyclists on an indoor velodrome, taking into account the effects of leaning in the bends and the actual position of the rider on the track using data collected by the SRM training system. This model uses forward integration to predict the velocity of the centre of mass and the finishing time for IP athletes, accurate to within 3% with currently available input data.

Keywords: Mathematical modelling; Individual pursuit; Cycling; Drag

1. Introduction

Mathematical modelling has been used to assess the performance of athletes in a number of different sports for more than a decade. A number of models have been developed for cycling alone, to assist athletes and coaches with training and performance by predicting the velocity and power output for a specific scenario [1-7]. The precision, accuracy and error of some of these models has been reviewed by González-Haro, Galilea, & Escanero [8] who concluded that the model developed by Olds et al. [7] had the highest precision and accuracy for estimating the peak power output in the velodrome compared to SRM™ data. However, this model was based on road-cycling, and did not take into account the effects of leaning in the bends. The only models specifically designed for track cycling have been developed by Lukes, Carré & Haake [9] and by Martin, et al. [3] which assume the track is circular. No published model has yet incorporated the real power output of the rider depending on their position on the track, which also takes accounts for leaning in the bends on an indoor velodrome.

1.1 The significance of the bends

When a cyclist goes around a bend, the inward (centripetal) acceleration, mV^2/R_T, must also be taken into account, as the cyclist is tilted over at an angle, ψ_L, from the vertical with his/her centre of mass following a trajectory with a smaller radius of curvature R_T than the radius of the bend (Fig.1). This leads to an expression for

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the velocity of the centre of mass, Eq.1, which can be applied at any position on the track. According to Kyle [10] there is also a 4% increase in tire rolling resistance in the straights and a 9.5% increase in tire rolling resistance in the bends due to the steering effect (slip angle) on a banked, velodrome track; even in the straights the track has a banking angle and the rider must steer to remain on track.

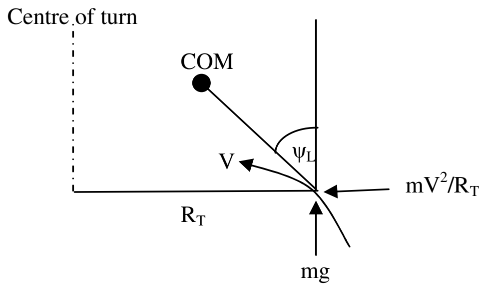

Fig.1. Free body diagram of a cyclist travelling around a bend. The velocity vector, V, acts into the plane of the paper.

COM is the centre of mass of the cyclist, R_T is the radius of curvature (m), V is the velocity of the wheels (m/s), m is the mass of the system (kg), g is the acceleration due to gravity (m/s^2), and ψ_L is the leaning angle (°).

The aim of this study was to develop a mathematical model for the Individual Pursuit (IP) on an indoor velodrome, taking into account the leaning effect in the bends and using real-life SRM™ power data, which varies according to the position of the athlete on the track.

2. Methods

2.1 Assumptions

The effects of wind and gradient were excluded from this model because they can be considered negligible on an indoor, symmetrical velodrome, assuming the athlete remains at the same height on the track throughout the race, as they do if they follow the black line (shortest path). The velocity of the air relative to the rider was assumed to be equal to wheel velocity due to the absence of wind. The actual power output of the athlete was reduced by 2% to account for drive-train loss and frame flexing, and the coefficient of rolling resistance of the tires and bearings were each taken to be 0.001 for a racing tire on a wooden track surface [11].The centre of mass of the cyclist was assumed to be the height of the seat from the ground, and the ratio of rider weight on the front and rear wheel taken to be 40% and 60% respectively. The rear wheel was also assumed to draft the front wheel, experiencing only 60% of the front wheel drag power. Tyre rolling resistance was increased by 9.5% in the bends and 4% in the straights due to the steering effect on a banked velodrome track [10].

2.2 Expressions for power supply and demand

A cyclist must do work in order to overcome all resistive forces acting on the bike and rider system, 96% of which comes from aerodynamic drag (Eq.1) [3]. Power is the rate of performing work, so the power lost due to aerodynamic drag is given by Eq.2. Other significant resistive forces acting on a bike and rider system include rolling resistance of the tires and bearings, weight resistance, and wheel drag; the power lost due to these resistive forces are shown in Eq. 3-5. The total power loss (P_{Loss}) is therefore the sum of all power lost to these resistive force, Eq.6.

$$D = \frac{1}{2}\rho C_d A V^2 \tag{1}$$

$$P_D = \frac{1}{2}\rho C_d A V^3 \tag{2}$$

$$P_{rr} = C_{rr}\left(N + \frac{mV^2}{R}\sin\psi_{\mathrm{L}}\right)V\cos\theta\cos\phi \tag{3}$$

$$P_g = mgV\sin\theta \tag{4}$$

$$P_w = \alpha V^2 + \beta V + \gamma \tag{5}$$

$$P_{Loss} = P_D + P_{rr} + P_g + P_w \tag{6}$$

where D is the aerodynamic drag (N), ρ is the air density (kg/m^3), C_d is the drag coefficient, A is the frontal area of the rider and bike frame (excluding the wheels) (m^2), V is the velocity of the wheels (m/s), P_{rr} is the power lost due to rolling resistance (W), C_{rr} is the coefficient of rolling resistance, N is the normal force acting on the tyre or bearing (N), m is the total mass of the system (kg), R is the radius of curvature (m), ψ_{L} is the lean angle from vertical (rad), θ is the grade angle of the track (rad), ϕ is the banking angle of the track (rad), P_g is the power lost due to weight resistance (W), g is the acceleration due to gravity, P_w is the power lost due to wheel resistance (W), and α, β, and γ are the coefficients used to fit a quadratic to the power absorbed by the drag of the wheel. Wheel drag data is taken from Jermy, Moore & Bloomfield [12]. The power/velocity relationship is expected to be a cubic, but a quadratic fits the data better, apart from returning some negative values at low speed.

The total power dissipated by the cyclist comprises the losses described above, plus the power absorbed by acceleration (Eq.7) which comprises translational and rotational acceleration the wheels, frame and rider (Eq.8 and Eq.9) and rotational or reciprocating acceleration of the legs (thighs, calves and feet) (Eq.10). The total power dissipated is equated to the instantaneous power output of the rider and the resulting equation solved to determine the acceleration a (m/s^2). For this model, instantaneous power was obtained from SRM data and reduced by 2% to account for drive-train loss.

$$P_{accelloss} = P_t + P_r + P_l \tag{7}$$

$$P_t = maV \tag{8}$$

$$P_r = aI\frac{\omega}{r} \tag{9}$$

$$P_l = aI\frac{C}{r_g} \tag{10}$$

where P_t is the power absorbed by translational acceleration of the wheels (W), m is the mass of the wheels (kg), V is the velocity of the wheels (m/s), P_r is the power absorbed by rotational acceleration of the wheels (W), I is the moment of inertia (kgm^2), r is the radius of the wheels, P_l is the power absorbed by acceleration of the thigh, foot or calf (W), C is the pedaling cadence (rpm), and r_g is the radius of gyration (m).

Additional expressions, which were incorporated into the model in order to refine the accuracy, include the leaning angle of the rider (Eq.11), the radius of curvature of the centre of mass (Eq.12), and the velocity of the centre of mass (Eq.13). The leaning angle of the centre of mass can be determined by equating vectors of weight (mg) and centripetal acceleration (mV^2/R_T) shown in Fig.1, as both the weight and centripetal force are proportional to mass, and their moments proportional to the height of the centre of mass. Acceleration occurs when the instantaneous power supplied by the system is greater than the power loss (Eq.14).

$$\psi = \tan^{-1}\left(\frac{V^2}{Rg}\right) \tag{11}$$

$$R_c = R - H\sin\left[\tan^{-1}\left(\frac{V^2}{Rg}\right)\right] \tag{12}$$

$$V_c = \frac{V}{R} \times R_c \tag{13}$$

$$a = \frac{P_{\text{instant}} - P_{\text{loss}}}{\text{mV}} \tag{14}$$

where ψ is the lean angle from vertical (rad), V is the velocity of the wheels (m/s), R is the radius of curvature of the wheel (m), g is the acceleration due to gravity (m/s), R_c is the radius of curvature of the centre of mass (m), H is the seat height (m), V_c is the velocity of the centre of mass, a is the acceleration of the system (m/s^2), $P_{Instant}$ is the instantaneous power taken from the SRM training system minus drive-train loss (W), P_{Loss} is the total power lost due to resistance (W), and P_{Supply} is the total power absorbed by acceleration of the wheels and legs (W).

2.3 Input parameters and algorithm

SRM power and velocity data were obtained from an elite, female, New Zealand cyclist for the 2008 Manchester World Championship 3000m IP event. The temperature, atmospheric pressure and relative humidity at the time of the race were also taken from the SRM training system, and the air density calculated using the method described by Davis [13]. Frame and wheel data were obtained from the manufacturer of the bike ridden for the race specified, and the height, weight, seat height and cadence of the athlete obtained from the coaches. The frontal area of the cyclist, bike and wheels was calculated from a photo using the digitizing method [14] and the C_dA determined from wind tunnel data. The frontal area of the wheels was determined and Eq. 15 used to determine the C_dA for the athlete and bike, excluding the wheels, in order to avoid including the wheel drag twice.

$$C_d A_{\text{rider+bike}} = C_d A_{\text{rider+bike+wheels}} \frac{A_{\text{rider+bike+wheels}} - A_{wheels}}{A_{\text{rider+bike+wheels}}} \tag{15}$$

where C_dA is the drag area, $A_{rider+bike+wheels}$ is the frontal area of the rider, bike and wheels found from the photo, and A(w) is the frontal area of the wheels calculated from the wheel geometry.

The actual distance travelled by the athlete for the race determined from video footage. Track data for the velodrome in Manchester, UK, was obtained, including the overall track length, grade angle, banking angle and radius of curvature throughout the bends and straights.

The initial acceleration of the athlete was determined using Eq.14, for an initial velocity of $2.778\text{x}10^{-6}\text{ms}^{-1}$ and initial distance of 0m at time=0. Once the initial acceleration had been determined, first order forward integration, with a 0.005s time-increment, was used to calculate the change in velocity, and hence the velocity profile, of the centre of mass of the athlete and the corresponding distance travelled. The estimated finishing time was taken to be the time at which the actual distance travelled by the athlete was reached.

3. Results

The actual finishing time for the athlete used as a subject for the model for the female, IP event at the 2008 Manchester World Championships was 215.212 seconds. The mathematical model predicted a finishing time of 209.744 seconds, underestimating the finishing time by 2.5%. The results in Fig.2a show that the model accurately predicts the pattern of velocity of the centre of mass of the athlete, where the velocity increases in the straights and decreases in the bends. However, the velocity data for the model and SRM training system are slightly offset for the first 100 seconds. This could be due to the accuracy of the SRM training system; data is not recorded until the first complete revolution of the wheel has been completed. Fig.2b shows the pattern of velocity of the centre of mass predicted by the model compared to the SRM velocity data with the start assumed to occur 3 seconds earlier. Although this improves the relationship between the model and SRM data for velocity for the first 100 seconds, the remaining data now appears to be offset.

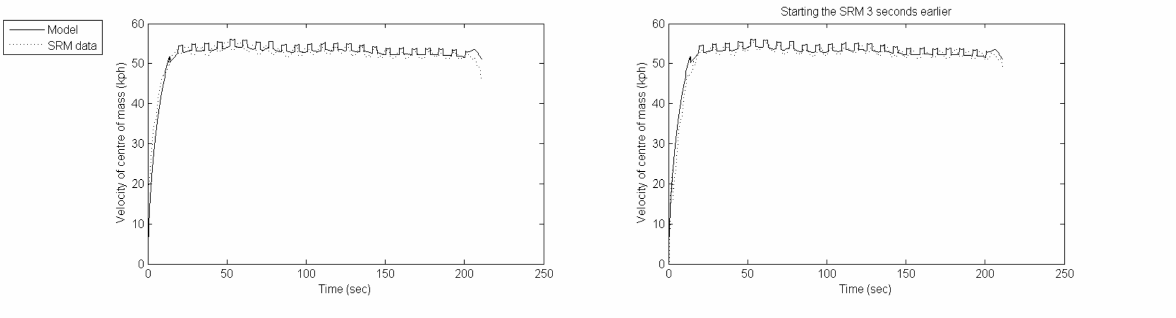

Fig.2. Comparison of the velocity of the centre of mass predicted by the model and the actual velocity recorded by the SRM training system (a) Original data; (b) Starting the SRM velocity data 3 seconds earlier.

4. Conclusion

This paper has presented a mathematical model for IP cyclists, based on real-life SRM power data, therefore taking into account the actual position of the athlete on the track. The effects of leaning angle and reduction in tire rolling resistance as the athlete cycles around the bends has been incorporated into the model, which results in an accurate prediction of the velocity of the centre of mass as the cyclists goes around the track. However, there is a slight offset between the velocity of the centre of mass predicted by the model and the actual velocity recorded by the SRM training system. This may be due to the accuracy of the SRM training system, which does not record data until the first revolution of the wheel has been completed.

The model presented predicts the finishing time for an elite, female, New Zealand, IP athlete to within a 3% error; the model underestimated the finishing time by 6 seconds. However, the accuracy of the model could be improved even further; negative values of the quadratic fitted wheel drag were observed at low velocities, which were eliminated by setting all the negative values to zero. These negative errors were due to fitting a quadratic to a set of data which should properly be a cubic, and which contains noise. The accuracy of the model is highly dependent on the product of drag and frontal area of the cyclist (C_dA). The effect of yaw has a strong influence on the C_dA, and although yaw on an indoor velodrome is negligible, it is not completely absent. The accuracy of this model could therefore be improved by measuring the actual yaw present for the conditions being simulated, and incorporating this into the demand side expressions.

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